Chapter 1. Simple Examples of Propagation.

This chapter presents examples of wave propagation governed by hyperbolic equations. The fundamental concepts introduced, propagation of singularities, group velocity, and short wavelength asymptotics is each the center piece of important mathematical developments. In addition we introduce the method of nonstationary phase which is a fundamental tool for estimating oscillatory integrals. The examples are elementary. They could each be part of an introductory course in partial differential equations, but usually are not. In sections 1.3, 1.5, 1.6 we derive in simple situations the three basis laws of physical geometric optics.

Wave like solutions of partial differential equations have spatially localized structures whose evolution in time can be followed. The most common are solutions with propagating singularities and solutions which are modulated wave trains. The latter have the form

$$a(t,x)\, e^{i\phi(t,x)/\epsilon}$$

with smooth profile a, real valued smooth phase, ϕ, with $d\phi \neq 0$ on the support of a. The parameter ϵ is small compared to the scale on which a and ϕ vary. Both involve radically different spatial scales. For the second it is the scale(s) on which the profile varies and the much smaller wavelength. The classic example is light with a wavelength on the order of 5×10^{-5} centimeter. Singularities are often restricted to varieties of lower codimension, hence of width equal to zero which is infinitely small compared to the scales of their other variations. Real world waves modeled by such solutions have the singular behavior spread over very small lengths, not exactly zero.

The path of a localized structure in space time is curvelike, and such curves are often called **rays**. When phenomena are described by partial differential equations, linking the above ideas with the equation means finding solutions whose salient features are localized and in simple cases are described by transport equations along rays. In the case of wavetrains, such results appear in an asymptotic analysis as $\epsilon \to 0$.

In this chapter some introductory examples are presented that illustrate propagation of singularities, propagation of energy, group velocity and short wavelength asymptotics. That energy and singularities may behave very differently is a consequence of the dichotomy that up to an error as small as one likes in energy, the data can be replaced by data with compactly supported Fourier transform. In contrast, up to an error as smooth as one likes the data can be replaced by data with Fourier transform vanishing on $|\xi| \leq R$ with R as large as one likes. Propagation of singularities is about short wavelengths while propagation of energy is about wavelengths bounded away from 0. When most the energy is carried in relatively short wavelengths, for example the wave packets above, the two tend to propagate in the same way.

§1.1. Examples of propagation of singularities using progressing waves.

D'Alembert's general solution of the one dimensional wave equation,

$$u_{tt} - u_{xx} = 0\,, \tag{1.1.1}$$

is the sum of progressing waves

$$f(x-t) + g(x+t)\,. \tag{1.1.2}$$

The rays are the integral curves of

$$\partial_t \pm \partial_x \,. \tag{1.1.3}$$

Structures are rigidly transported at speeds ± 1. The transport equation expresses constancy on the integral curves of (1.1.3).

There is an energy law for solutions suitably small at infinity,

$$\int_{\mathbb{R}} u_t^2 + u_x^2 \, dx \;=\; \text{independent of time}\,.$$

This is proved by differentiating the energy with respect to time and using an integration by parts to show that the resulting expression vanishes when (1.1.1) is satisfied.

The fundamental solution which solves (1.1.3) together with the initial values

$$u(0,x) = 0\,, \qquad u_t(0,x) = \delta(x)\,, \tag{1.1.4}$$

is given by the explicit formula

$$u(t,x) \;=\; \frac{\operatorname{sgn} t}{2}\,\chi_{[-t,t]} \;=\; \frac{1}{2}\,\Big(h(x-t) - h(x+t)\Big)\,, \tag{1.1.5}$$

where h denotes Heaviside's function, the characteristic function of $]0,\infty[$ Note the singularities which propagate to the left and the right.

Interesting things happen if one adds a lower order term. For example, consider the Klein-Gordon equation

$$u_{tt} - u_{xx} + u = 0\,. \tag{1.1.6}$$

In sharp contrast with (1.1.2), there are hardly any undistorted progressing wave solutions.

Exercise 1. *Find all solution of (1.1.6) of the form $f(x-ct)$.* **Discussion.** *The special solutions of the form $e^{i(\tau t - x\xi)}$ with $\xi \in \mathbb{R}$ are particularly important since the general solution is a Fourier superposition of these special plane waves. The equation $\tau = \tau(\xi)$ defining such solutions is called the* **dispersion relation** *of (3.1.6).*

There is an energy conservation law. Denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing smooth functions. That is, functions such that for all α, β

$$\sup_{x\in\mathbb{R}^d} \left| x^\beta \partial_x^\alpha \psi(x) \right| \;<\; \infty\,.$$

Exercise 2. *Prove that if $u \in C^\infty(\mathbb{R} : \mathcal{S}(\mathbb{R}))$ is a real valued solution of the Klein-Gordon equation, then*

$$\int u_t^2 + u_x^2 + u^2 \; dx$$

is independent of t. This quantity is called the **energy** *and is denoted E.*

The fundamental solution, that is the solution with initial data (1.1.4), is not as simple as in the case of the wave equation. However, the singularities can be exactly computed. Introduce

$$h_n(x) \ := \ \begin{cases} x^n/n! & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x \leq 0 \end{cases} . \tag{1.1.7}$$

Then

$$\frac{d}{dx}\, h_{n+1} = h_n \,, \qquad \text{for} \quad n \geq 0\,. \tag{1.1.8}$$

Exercise 3. *Show that there are uniquely determined functions* $a_n(t)$ *satisfying*

$$a_0(0) = 1/2, \quad \text{and} \quad a_n(0) = 0 \quad \text{for} \quad n \geq 1\,,$$

and so that for all $N \geq 2$,

$$\Big(\partial_t^2 - \partial_x^2 + 1\Big) \sum_{n=0}^{N} a_n(t)\, h_n(x-t) \in C^{N-2}(\mathbb{R}^2)\,. \tag{1.1.9}$$

In this case, we say that the series

$$\sum_{n=0}^{\infty} a_n(t)\, h_n(t-x)$$

is a formal solution of $(\partial_t^2 - \partial_x^2 + 1)u \in C^\infty$. **Hint.** *Pay special attention to the most singular term(s).*

Exercise 4. *Suppose that* u *is the fundamental solution of the Klein-Gordon equation and* $M \geq 0$. *Find a distribution* w_M *such that* $u - w_M \in C^M(\mathbb{R}^2)$. *Show that the fundamental solution of the wave equation and that of the Klein-Gordon equation differ by a Lipschitz continuous function. Show that the singular supports of the two fundamental solutions are equal.* **Hint** *Add (1.1.9) to its spatial reflection.*

Exercise 5. *Study the fundamental solution for the dissipative wave equation*

$$u_{tt} - u_{xx} + 2u_t = 0\,. \tag{1.1.10}$$

In particular show that it is no longer a continuous perturbation of that for the wave equation, but nevertheless the singular support agrees with that of the wave equation. **Hint.** *Seek solutions of* $(\partial_t^2 - \partial_x^2 + 2\partial_t)u \in C^\infty$ *of the form* $\sum_n b_n(t)\, h_n(t-x)$.

The method in the above exercises is called **progressing wave expansions**. It is discussed in more generality in chapter 6 of Courant-Hilbert Vol. 2, and in Lax's *Lectures on Hyperbolic Partial Differential Equations.* The higher dimensional analogue of these solutions are singular along codimension one characteristic hypersurfaces in space time.

The singularities propagate satisfying transport equations along rays lying in the hypersurface. The general class goes under the name *conormal solutions*. They are discussed, for example, in M. Beals' book cited in the references. In practice they describe propagating wavefronts. Luneberg [Lun] early on recognized that the propagation laws for fronts of singularities coincide with the classical laws of geometric optics.

§1.2. Group velocity and the method of nonstationary phase.

The Klein-Gordon equation, can be solved explicitly using the Fourier transform. The computation of the singularities of the fundamental solution of the Klein-Gordon equation suggests that the main part of solutions travel with speed equal to 1. One might expect that the energy in a disk growing linearly in time at a speed slower than one would be small. For compactly supported data, such a disk would contain no singularities for large time. Thus it is not unreasonable to guess that for any $\sigma < 1$, and $R > 0$

$$\limsup_{t\to\infty} \int_{|x|<R+\sigma t} u_t^2 + u_x^2 + u^2\, dx \;=\; 0. \tag{1.2.1}$$

If (1.2.1) were true for a dense set in the energy space it would follow for all by uniform boundedness.

The energy method shows that speeds are no larger than one. The idea about the main part of the solution expressed in (1.2.1) is dead wrong for the Klein-Gordon equation. The main part of the energy travels strictly slower than speed 1, even though singularities travel with speed exactly equal to 1.

The solutions of the Klein-Gordon equation,

$$u_{tt} \;-\; \Delta\, u \;+\; u \;=\; 0\,, \qquad (t,x)\;\in\;\mathbb{R}^{1+d}\,,$$

take the form

$$u \;=\; \sum_{\pm} (2\pi)^{-d/2} \int a_\pm(\xi)\; e^{i(\pm\langle\xi\rangle t + x.\xi)}\; d\xi\,, \qquad \langle\xi\rangle := \big(1+|\xi|^2\big)^{1/2},$$

$$\hat u(0,\xi) \;=\; a_+(\xi)+a_-(\xi)\,, \qquad \hat u_t(0,\xi) \;=\; i\,\langle\xi\rangle\,\big(a_+(\xi)-a_-(\xi)\big)\,.$$

The energy is equal to

$$\int u_t^2 \;+\; |\nabla_x u|^2 \;+\; u^2\; dx \;=\; \int \langle\xi\rangle^2\,\big(|a_+(\xi)|^2 \;+\; |a_-(\xi)|^2\big)\; d\xi\,.$$

Consider the behavior for large times. The phases $\phi_\pm(t,x,\xi) = \pm\langle\xi\rangle t + x.\xi$ have gradients

$$\nabla_\xi \phi_\pm(t,x,\xi) \;:=\; \nabla_\xi\Big(\pm\langle\xi\rangle t + x.\xi\Big) \;=\; \frac{\pm t\xi}{\langle\xi\rangle} + x \;=\; t\Big(\frac{\pm\xi}{\langle\xi\rangle} + \frac{x}{t}\Big)\,.$$

Thus at space time points (t,x) so that $t >> 1$ and

$$\frac{\pm\xi}{\langle\xi\rangle} + \frac{x}{t} \;\neq\; 0\,,$$

the phase oscillates rapidly and the contribution to the integral is expected to be small. Thus the contribution to the $a_\pm$ integral from $\xi \sim \underline{\xi}$ is felt predominantly at points where $x/t \sim \mp\underline{\xi}/\langle\underline{\xi}\rangle$. Setting $\tau_\pm(\xi) := \pm\langle\xi\rangle$ one has

$$\frac{\mp\underline{\xi}}{\langle\underline{\xi}\rangle} \;=\; -\nabla_\xi \tau_\pm(\underline{\xi})\,.$$

This recovers the formula for the group velocity introduced on purely geometric grounds in §2.4.

For $t \to \infty$ the contributions of the plane waves $a_\pm(\xi)e^{i(\tau_\pm(\xi)t+x,\xi)}$ with $\xi \sim \underline{\xi}$ are expected to be felt at points with $x/t \sim -\nabla_\xi \tau_\pm(\underline{\xi})$. A precise version is proved using the method of nonstationary phase.

Proposition 1.2.1. *Suppose that* $a_\pm(\xi) \in C_0^\infty(\mathbb{R}^d)$ *and define*

$$\mathbf{V} \;:=\; \big\{\mathbf{v} \;:\; \mathbf{v} = -\nabla_\xi \tau_\pm(\xi) \;\; \text{for some } \xi \in \operatorname{supp} \mathrm{a}_\pm\big\}$$

as the compact set of group velocities that appear in the plane wave decomposition of u. *Suppose that* $\mathbf{K} \subset \mathbb{R}^d \setminus \mathbf{V}$ *is closed and denote by* Γ *the cone*

$$\Gamma \;:=\; \Big\{(t,x) \;:\; t > 0, \;\; \text{and} \;\; x/t \in \mathbf{K}\Big\}\,.$$

Then for all $N > 0$, *there is a constant* C_N *so that for all* $|\alpha| \le N$.

$$(1 + t + |x|)^N \; \partial^\alpha_{t,x} u(t,x) \in L^\infty(\Gamma)\,.$$

Proof. The solution u is smooth with values in $\mathcal{S}$ so one need only consider $\{t \ge 1\}$. We estimate the $+$ summand. The $-$ summand can be treated similarly.

Introduce the first order differential operator

$$\ell(t,x,\partial) \;:=\; \frac{1}{i|\nabla_\xi \phi|^2} \, \sum_j \frac{\partial \phi}{\partial \xi_j} \, \frac{\partial}{\partial \xi_j}\,.$$

The coefficients are smooth functions on a neighborhood of Γ, and are homogeneous of degree minus one in (t,x) and satisfy

$$\frac{1}{|\nabla_\xi \phi|^2} \, \Big|\frac{\partial \phi}{\partial \xi_j}\Big| \;\le\; C(t+|x|)^{-1} \quad \text{for} \quad (t,x,\xi) \in \Gamma \times \; \operatorname{supp} a_+\,.$$

In addition,

$$\ell(t,x,\partial_\xi)\, e^{i\phi} \;=\; e^{i\phi}\,.$$

Therefore

$$\int a_+(\xi)\; e^{i\phi}\; d\xi = \int a_+(\xi)\; \ell^N e^{i\phi}\; d\xi\,.$$

Denote by $\ell^\dagger$ the transpose of ℓ and integrate by parts to find

$$\int a_+(\xi)\; e^{i\phi}\; d\xi = \int [(\ell^\dagger)^N a_+(\xi)]\; e^{i\phi}\; d\xi\,.$$

The operator

$$(\ell^\dagger)^N \;=\; \sum_{|\alpha|\le N} c_\alpha(t,x,\xi)\,\partial_\xi^\alpha$$

with coefficients c_α smooth on a neighborhood of Γ, homogeneous of degree $-N$ in t, x, with

$$|c_\alpha(t,x)| \;\le\; C(\alpha)(t+|x|)^{-N} \quad \text{for} \quad (t,x,\xi)\in\Gamma\times\;\operatorname{supp} a_+\,.$$

It follows that

$$\Big|\int a_+(\xi)\; e^{i\phi}\; d\xi\Big| \;\le\; C\,(t+|x|)^{-N}\,.$$

Since the t, x derivatives of this integral are again integrals of the same form, this suffices to prove the proposition. ∎

In particular, if $\tilde{\mathbf{V}}\supset\mathbf{V}$ is a compact neighborhood of $\mathbf{V}$, then for $t\to\infty$ virtually all the energy of a solution is contained in the cone $\big\{(t,x)\,:\,x/t\in\tilde{\mathbf{V}}\big\}$. This is particularly interesting when $a_\pm$ are supported in a small neighborhood of a fixed $\underline{\xi}$. Then for large times virtually all the energy is localized in two thin cones containing the lines $x=-t\,\nabla_\xi\tau_\pm(\underline{\xi})$ traveling with the group velocities associated to $\underline{\xi}$.

The integration by parts method introduced in this proof is very important. The next estimate for nonstationary oscillatory integrals is a straight forward application. The fact that the estimate is uniform in the phases is useful.

Lemma of Nonstationary Phase 1.2.2. *Suppose that Ω is a bounded open subset of $\mathbb{R}^d$ and that $C_1>1$. Then there is a constant $C_2>0$ so that for all $\phi\in C^m(\Omega\,;\mathbb{R})$ so that*

$$\forall|\alpha|\le m\,,\quad \|\partial^\alpha\phi\|_{L^\infty}\le C_1\,,\quad \text{and,}\quad \forall x\in\Omega\,,\quad C_1^{-1}\;\le|\nabla_x\phi|\;\le\;C_1\,,$$

and $\forall f\in C_0^m(\Omega)$,

$$\Big|\int\; e^{i\phi/\epsilon}\; f(x)\; dx\Big| \;\le\; C_2\,\epsilon^m\sum_{|\alpha|\le m}\|\partial^\alpha f\|_{L^1}\,.$$

Exercise 6. *Prove the Lemma.* **Hint.** Use

$$\ell(x,\partial_x)\;:=\;\frac{\nabla_x\phi}{i\,|\nabla_x\phi|}\,.\,\partial_x\,.$$

A special case are the phases $\phi = x.\xi$ with ξ belonging to a compact subset of $\mathbb{R}^d \setminus 0$. The Lemma is then equivalent to the rapid decay of the Fourier transform of smooth compactly supported functions. That decay is proved by integration by parts. The general result can be reduced to the special case of the Fourier transform. Since the gradient of ϕ does not vanish, for each $\underline{x} \in \operatorname{supp} f$ there is a neighborhood and a nonlinear change of coordinates so that in the new coordinates ϕ is equal to x_1. Using a partition of unity, one can suppose that f is the sum of a finite number of functions each supported in one of the neighborhoods. For each such function, a change of coordinates yields an integral of the form

$$\int e^{ix_1/\epsilon}\, g(x)\, dx \;=\; c\,\hat{g}(1/\epsilon, 0, \ldots, 0)\,,$$

which is rapidly decaying since it is the transform of an element of $C_0^\infty(\mathbb{R}^d)$.

Exercise 7. *Suppose that $f \in H^1(\mathbb{R})$ and $g \in L^2(\mathbb{R})$ and that u is the unique solution of the Klein-Gordon equation with initial data*

$$u(0,x) = f(x)\,, \qquad u_t(0,x) = g(x)\,. \tag{1.2.2}$$

Prove that for any $\epsilon > 0$ and $R \geq 0$, there is a $\delta > 0$ so that

$$\limsup_{t\to\infty} \int_{|x|>(1-\delta)t-R} u_t^2 + u_x^2 + u^2\, dx \;<\; \epsilon\,. \tag{1.2.3}$$

Hint. *Replace $\hat{f}, \hat{g}$ by compactly support smooth functions making an error at most $\epsilon/2$ in energy. Then use the above proposition noting that the group velocities are uniformly smaller than one on the supports of $a_\pm$.*

Discussion. *Note that as $\xi \to \infty$ the group velocities approach ± 1. Thus high frequencies will propagate at speeds nearly equal to one. In particular they travel at the same speed. High frequency signals stay together better than low frequency signals. Since singularities of solutions are made of only the high frequencies (modifying the data by an element of $\mathcal{S}$ modifies the solution by such an element and therefore by a smooth term) one expects singularities to propagate at speeds ± 1 which is exactly what is true for the fundamental solution. Once known for the fundamental solution it follows for all. The simple proof is an exercise in my book [R?], pages 164-165.*

The analysis of Exercise 7 does not apply to the fundamental solution since the latter does not have finite energy. However it belongs to $C^j(\mathbf{R} : H^{s-j}(\mathbb{R}))$ for all $j \in \mathbb{N}$ and $s < 1/2$. Thus the next result provides a good replacement of (1.2.3).

Exercise 8. *Suppose that u is the fundamental solution of the Klein-Gordon equation (1.1.6) and that $s < 1/2$. If $0 \leq \chi \in C^\infty(\mathbb{R})$ is a plateau cutoff supported on the positive half line, that is*

$$\chi(x) = 0 \quad \text{for} \quad x \leq 0 \qquad \text{and} \quad \chi(x) = 1 \quad \text{for} \quad x \geq 1\,,$$

then for all $R \geq 0$,

$$\lim_{t\to\infty} \| \chi(R+|x|-t)\, u(t,x) \|_{H^s(\mathbb{R}_x)} = 0\,. \tag{1.2.4}$$

Hint. *Prove that*

$$\| \chi\, u(t) \|_{H^s(\mathbb{R})} \leq C\Big(\|u(0)\|_{H^s(\mathbb{R})} + \|u_t(0)\|_{H^{s-1}(\mathbb{R})} \Big)$$

with C *independent of* t *and the initial data. Conclude that it suffices to prove (3.2.4) with initial data* $u(0), u_t(0)$ *dense in* $H^s \times H^{s-1}$*. Take the dense set to be data with Fourier Transform in* $C_0^\infty(\mathbb{R})$*.*

Discussion. *This shows that though the singularities move at speed* ± 1*, the energy moving at this speed is negligible in the limit* $t \to \infty$*.*

These examples illustrate the important observation that the propagation of singularities in solutions and the propagation of the majority of the energy may be governed by different rules. For the Klein Gordon equation at least, both answers can be determined from considerations of group velocities.

§1.3. Fourier synthesis and rectilinear propagation.

For equations with constant coefficients, solutions of the initial value problem are expressed as Fourier integrals. Injecting short wavelength initial data and performing an asymptotic analysis yields the approximations of geometric optics. This is how such approximations were first justified in the nineteenth century. It is also the motivating example for the more general theory. The short wavelength approximations explain the *rectilinear propagation of waves* in homogeneous media. This is the first of the three basic physical laws of geometric optics. It explains, among other things, the formation of shadows.

Consider the initial value problem

$$\Box u \;:=\; u_{tt} - \Delta\, u \;:=\; \frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2} \;=\; 0\,, \qquad u(0,x) = f\,, \quad u_t(0,x) = g\,. \tag{1.3.1}$$

Fourier transformation with respect to the x variables yields

$$\partial_t^2 \hat{u}(t,\xi) \;+\; |\xi|^2\, \hat{u}(t,\xi) \;=\; 0\,, \qquad \hat{u}(0,\xi) = \hat{f}\,, \quad \partial_t \hat{u}(0,\xi) = \hat{g}\,.$$

Solve the ordinary differential equations in t to find

$$\hat{u}(t,\xi) \;=\; \hat{f}(\xi)\, \cos t|\xi| + \hat{g}(\xi)\, \frac{\sin t|\xi|}{|\xi|} \;=\; a_+(\xi)\, e^{i(x\xi - t|\xi|)} - a_-(\xi)\, e^{i(x\xi + t|\xi|)}\,, \tag{1.3.2}$$

with

$$2\, a_+ \;=\; \hat{g} \;-\; \frac{\hat{f}}{i|\xi|}\,, \qquad 2\, a_+ \;=\; \hat{g} \;+\; \frac{\hat{f}}{|\xi|}\,. \tag{1.3.3}$$

The right hand side of (1.3.2) is an expression in terms of the plane waves $e^{i(x\xi \mp t|\xi|)}$ with amplitudes $a_\pm(\xi)$ and dispersion relations $\tau = \mp|\xi|$. The group velocities associated to $a_\pm$ are

$$\mathbf{v} \;=\; -\nabla_\xi \tau \;=\; -\nabla_\xi(\mp|\xi|) \;=\; \pm\frac{\xi}{|\xi|}\,.$$

The last expression motivates the sign convention for $a_\pm(\xi)$.

The solution is the sum of two terms

$$u^\epsilon_\pm(t,x) \;:=\; \frac{1}{(2\pi)^{d/2}}\int a_\pm(\xi)\; e^{i(x\xi\mp t|\xi|)}\,d\xi\,.$$

The natural conserved energy for the wave equation is equal to

$$\frac{1}{2}\int |u_t(t,x)|^2 \;+\; |\nabla_x u(t,x)|^2\;dx \;=\; \int |\xi|^2\big(|a_+(\xi)|^2 \;+\; |a_-(\xi)|^2\big)\;d\xi\,.$$

There are conservations of all orders,

$$\frac{1}{2}\,\|\nabla_{t,x}u(t)\|^2_{H^s(\mathbb{R}^d)} \;=\; \int \langle\xi\rangle^{2s}\;|\xi|^2\;\big(|a_+(\xi)|^2 \;+\; |a_-(\xi)|^2\big)\;d\xi\,.$$

Take initial data oscillating with wavelength of order ϵ and linear phase equal to x_1/ϵ,

$$u^\epsilon(0,x) \;=\; g(x)\,e^{ix_1/\epsilon}\,,\qquad u^\epsilon_t(0,x) \;=\; 0\,,\qquad g\;\in\;\cap_s H^s(\mathbb{R}^d)\,. \tag{1.3.4}$$

The choice $u_t = 0$ postpones dealing with the factor $1/|\xi|$ in (1.3.3). Applying (1.3.3) with $f = 0$ and with

$$\hat u(0,\xi) \;=\; \mathcal{F}\big(g(x)\,e^{ix_1/\epsilon}\big) \;=\; \hat g(\xi - \mathbf{e}_1/\epsilon)\,,$$

yields

$$u^\epsilon_\pm(t,x) \;:=\; \frac{1}{(2\pi)^{d/2}}\int a_\pm(\xi - \mathbf{e}_1/\epsilon)\; e^{i(x\xi\mp t|\xi|)}\,d\xi\,.$$

Analyse u^ϵ_+, the other being entirely analogous. For ease of reading the subscript plus is omitted. Introduce $\zeta := \xi - \mathbf{e}_1/\epsilon$,

$$u^\epsilon(t,x) \;=\; \frac{1}{(2\pi)^{d/2}}\int a(\zeta)\; e^{ix(\mathbf{e}_1+\epsilon\zeta)/\epsilon}\; e^{-it|\mathbf{e}_1+\epsilon\zeta|/\epsilon}\;d\zeta\,. \tag{1.3.5}$$

The approximation of geometric optics comes from injecting the first order Taylor expansion,

$$\big|\mathbf{e}_1 + \epsilon\zeta\big| \;\approx\; 1 + \epsilon\zeta_1\,.$$

Collecting the rapidly oscillating terms $e^{i(x-t)/\epsilon}$ which do not depend on ζ yields

$$u^\epsilon(t,x) \approx u_{\text{approx}} := e^{i(x_1-t)/\epsilon}\;a(t,x),\quad a(t,x) := \frac{1}{(2\pi)^{d/2}}\int a(\zeta)\,e^{i(x\zeta - t\zeta_1)}\,d\zeta\,. \tag{1.3.6}$$

Write $x - t\zeta_1 = (x - t\mathbf{e}_1).\zeta$ to find,

$$a(t,x) = h(x - t\mathbf{e}_1) = h(x_1 - t, x_2, \dots, x_d)\,, \qquad h(x) := \frac{1}{(2\pi)^{d/2}} \int a(\zeta)\, e^{ix\zeta}\, d\zeta\,.$$

The approximation is a wave translating rigidly at speed one along the x_1 axis. The waveform h is arbitrary. The high frequency oscillation in the data yields a solution which to leading order resembles the collumnated light from a flashlight. If the support of h is small the approximate solution resembles a light ray.

The amplitude a satisfies the transport equation

$$\frac{\partial a}{\partial t} + \frac{\partial a}{\partial x_1} \;=\; 0\,,$$

so is constant on the **rays** $\underline{x} + t\mathbf{e}_1$. This is the degree of simplification that is typical in geometric optics. The construction of a family of short wavelength approximate solutions of D'Alembert's wave equations required only the solutions of a simple transport equation.

The dispersion relation of the family of plane waves

$$e^{i(x.\xi + \tau t)} \;=\; e^{i(x.\xi - |\xi| t)}$$

is $\tau = -|\xi|$. The velocity of transport, $\mathbf{v} = (1, 0, \dots, 0)$, is the group velocity $\mathbf{v} = -\nabla_\xi \tau(\underline{\xi})$ at $\underline{\xi} = (1, 0, \dots, 0)$. The propagation is along the rays $x = \underline{x} + t\mathbf{e}_1$. For the opposite choice of sign the dispersion relation is $\tau = |\xi|$, and the rays are the lines $x = \underline{x} - t\mathbf{e}_1$.

Had we taken data with oscillatory factor $e^{ix.\xi/\epsilon}$ then the propagation would be at the group velocity $\pm\xi/|\xi|$.

Exercise. *The approximate solution (1.3.6) is a function $H(x - t\mathbf{e}_1)$ with $H(x) = e^{ix_1/\epsilon}\, h(x)$. Prove that $u = K(x - \mathbf{e}_1 t)$ is a solution of $\Box u = 0$ if and only if K is independent of $x_2, \dots, x_d$.* **Discussion. 1.** When the dimension $d > 1$, there are no non constant exact solutions $K(x - \mathbf{e}_1 t)$ with finite energy. **2.** The problem is purposely vague about hypotheses on K. The computation is algebraic. The conclusion is valid for $K \in C^2$ or $K \in L^2$ or K a distribution in x. **3.** When $d > 1$, the distribution $\delta(x - \mathbf{e}_1 t)$ is **not** a solution of the wave equation. The most obvious candidates to explain rectilinear rays are not solutions.

Next, analyse the error in (1.3.6). The first step is to extract the rapidly oscillating factor in (1.3.5) to define an exact amplitude $a^\epsilon_{\text{exact}}$,

$$u^\epsilon(t,x) = e^{i(x_1 - t)/\epsilon}\, \frac{1}{(2\pi)^{d/2}} \int a(\zeta)\; e^{ix.\zeta}\; e^{-it(|\mathbf{e}_1 + \epsilon\zeta| - 1)/\epsilon}\, d\zeta \;:=\; e^{i(x_1 - t)/\epsilon}\; a^\epsilon_{\text{exact}}\,. \quad (1.3.7)$$

Proposition 1.3.1. *The exact and approximate solutions of $\Box u^\epsilon = 0$ with Cauchy data (1.3.4) are given by*

$$u^\epsilon = \sum_{\pm} e^{i(x_1 \mp t)/\epsilon}\; a^\epsilon_{\text{exact},\pm}\,, \qquad u^\epsilon_{\text{approx}} = \sum_{\pm} e^{i(x_1 \mp t)/\epsilon}\; a_\pm(t,x)\,,$$

as in (1.3.7) and (1.3.6). The error is $O(\epsilon)$ on bounded time intervals. Precisely, there is a constant $C > 0$ so that for all s, ϵ, t,

$$\left\| a^\epsilon_{\text{exact},\pm} - a_\pm \right\|_{H^s(\mathbb{R}^N)} \;\leq\; C\,\epsilon\,\langle t\rangle\,\left\| \langle\zeta\rangle^{s+2}\; a_\pm(\zeta)\right\|_{L^2(\mathbb{R}^d)}\,.$$

Proof. It suffices to estimate the error with the plus sign. In that case, suppressing the subscript, the definitions yield

$$a^\epsilon_{\text{exact}} - a \;=\; \frac{1}{(2\pi)^{d/2}} \int a(\zeta)\; e^{ix.\zeta}\; \left(e^{-it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon)} - e^{-it\zeta_1}\right)\, d\zeta\,.$$

The definition of the $H^s(\mathbb{R}^d)$ norm yields

$$\left\| a^\epsilon_{\text{exact}} - a \right\|_{H^s(\mathbb{R}^N)} \;=\; \left\| \langle\zeta\rangle^s\; a(\zeta)\; \left(e^{-it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon} - e^{-it\zeta_1}\right)\right\|_{L^2(\mathbb{R}^N)}.$$

Taylor expansion yields for $|\beta| \leq 1/2$,

$$|\,\mathbf{e}_1 + \beta\,| \;=\; 1 + \beta_1 + r(\beta)\,, \qquad |r(\beta)| \leq C\,|\beta|^2\,.$$

Increasing C if needed, the same inequality is true for $|\beta| \geq 1/2$ as well.

Applied to $\beta = \epsilon\zeta$ this yields,

$$\Big| t\big(\big|\mathbf{e}_1 + \epsilon\zeta\big| - 1\big)/\epsilon - \zeta_1 x_1 \Big| \;\leq\; C\,\epsilon\,|t|\,|\zeta|^2\,,$$

so

$$\left| e^{-it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon} \;-\; e^{-it\zeta_1}\right| \;\leq\; C\,\epsilon\,|t|\,|\zeta|^2\,.$$

Therefore

$$\left\| \langle\zeta\rangle^s\; a(\zeta)\; \left(e^{-it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon} - e^{-it\zeta_1}\right)\right\|_{L^2(\mathbb{R}^d)} \;\leq\; C\,\epsilon\,|t|\,\left\| \langle\zeta\rangle^s |\zeta|^2\; a(\zeta)\right\|_{L^2}. \qquad (1.3.8)$$

Combining (1.3.7-1.3.8) yields the estimate of the Proposition. ∎

The approximation retains some accuracy so long as $t = o(1/\epsilon)$.

The approximation has the following geometric interpretation. One has a superposition of plane waves $e^{i(x\xi+t|\xi|)}$ with $\xi = (1/\epsilon, 0, \ldots, 0) + O(1)$. Replacing ξ by $(1/\epsilon, 0, \ldots, 0)$ and $|\xi|$ by $1/\epsilon$ in the plane waves yields the approximation (1.3.6).

The wave vectors, ξ, make an angle $O(\epsilon)$ with $\mathbf{e}_1$. The corresponding rays have velocities which differ by $O(\epsilon)$ so the rays remain close for times small compared with $1/\epsilon$. For longer times the fact that the group velocities are not parallel is important. The wave begins to spread out. Parallel group velocities is a reasonable approximation for times $t = o(1/\epsilon)$.

The example reveals several scales of time. For times $t << \epsilon$, u and its gradient are well approximated by their initial values. For times $\epsilon << t << 1$ $u \approx e^{i(x-t)/\epsilon} a(0,x)$. The solution begins to oscillate in time. For $t = O(1)$ the approximation $u \approx a(t,x)\, e^{i(x-t)/\epsilon}$ is appropriate. For times $t = O(1/\epsilon)$ the approximation ceases to be accurate. The more refined approximations valid on this longer time scale are called *diffractive geometric optics*. The reader is referred to [JMR, Diffr. GOP] for a treatment in the spirit of Chapters 7-8.

It is both typical and somewhat surprising of the approximations of geometric optics, that

$$\Box\big(u_{\rm approx} - u_{\rm exact}\big) \;=\; \Box u_{\rm approx} = O(1)\,,$$

is not small. The error $u_{\rm approx} - u_{\rm exact} = O(\epsilon)$ is smaller by a factor of ϵ. The residual $\Box u_{\rm approx}$ is rapidly oscillatory, so applying $\Box^{-1}$ gains the factor ϵ.

The analysis just performed can be carried out without fundamental change for initial oscillations with nonlinear phase. A nice description including the phase shift on crossing a focal point can be found in §12.2 of [Hö2].

Taylor expansion to higher order yields,

$$|\mathbf{e}_1 + \beta| \;\sim\; 1 + \beta_1 + \sum_{|\alpha|\ge 2} c_\alpha \beta^\alpha\,, \tag{1.3.9}$$

and

$$\big(|\mathbf{e}_1 + \epsilon\zeta| - 1\big)/\epsilon \sim \zeta_1 + \sum_{|\alpha|\ge 2} \epsilon^{|\alpha|-1}\, c_\alpha\, \zeta^\alpha, \qquad e^{it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon} \sim e^{it\zeta_1}\Big(1 + \sum_{|\gamma|\ge 1} \epsilon^\gamma\, h_\gamma(t,\zeta)\Big)\,.$$

Injecting in the formula for u^ϵ yields an expansion

$$a^\epsilon_{\rm exact} \;\sim\; a(t,x) + \epsilon\, a_1(t,x) + \epsilon^2 a_2(t,x) + \cdots\,, \tag{1.3.10}$$

$$a_j \;=\; (2\pi)^{-d/2} \int a(\zeta)\; e^{i(x\zeta - t\zeta_1)}\; h_j(t,\zeta)\; d\zeta\,. \tag{1.3.11}$$

The series is asymptotic as $\epsilon \to 0$ in the sense of Taylor series. For any s, N, truncating the series after N terms yields an approximate amplitude which differs from $a^\epsilon_{\rm exact}$ by $O(\epsilon^{N+1})$ in H^s uniformly on compact time intervals.

From (1.3.11) one computes

$$\big(\partial_t + \partial_1\big) a_j \;=\; (2\pi)^{-d/2} \int a(\zeta)\; e^{i(x\zeta - t\zeta_1)}\; \frac{\partial h_j}{\partial t}(t,\zeta)\; d\zeta \;=\; \frac{\partial h_j}{\partial t}\Big(t, \frac{1}{i}\partial_x\Big) a_0\,. \tag{1.3.12}$$

In particular, if the Cauchy data are supported in a set $\mathcal{O}$, then the amplitudes a_j are all supported in the tube of rays

$$\mathcal{T} \;:=\; \Big\{(t,x)\;:\; x = \underline{x} + t\mathbf{e}_1, \quad \underline{x} \in \mathcal{O}\Big\}\,.$$

Exercise. *Compute the first corrector* a_1.

Warning. Though the a_j are supported in this tube, it is not true that $a^\epsilon_{\rm exact}$ is supported in the tube.

To analyse the oscillatory initial value problem with $u(0) = 0$, $u_t(0) = f(x)\, e^{ix_1/\epsilon}$ requires one more idea to handle the contributions from $\xi \approx 0$ in the expression

$$u(t,x) \;=\; (2\pi)^{-d/2} \int \frac{\sin t|\xi|}{|\xi|}\; \hat f\Big(\xi - \frac{\mathbf{e}_1}{\epsilon}\Big)\; e^{ix\xi}\; d\xi\,.$$

Choose $\chi \in C_0^\infty(\mathbb{R}^d_\xi)$ with $\chi = 1$ on a neighborhood of $\xi = 0$. The cutoff integrand is equal to

$$\chi(\xi)\; \frac{\sin t|\xi|}{|\xi|}\; \frac{1}{\langle \xi - \mathbf{e}_1/\epsilon\rangle^s}\; k_s(\xi - \mathbf{e}_1/\epsilon)\; e^{ix\xi}\,, \qquad k_s(\xi) \;:=\; \langle\xi\rangle^s\; f(\xi) \;\in\; L^2(\mathbb{R}^d_\xi)\,.$$

Then

$$\Big\| \chi(\xi)\; \frac{\sin t|\xi|}{|\xi|}\; \frac{1}{\langle \xi - \mathbf{e}_1/\epsilon\rangle^s} \Big\|_{L^\infty(\mathbb{R}^d)} \;\le\; C_s\, |t|\, \epsilon^s\,, \qquad 0 < \epsilon \le 1\,.$$

It follows that

$$\Big\| \chi(\xi)\; \frac{\sin t|\xi|}{|\xi|}\; \frac{1}{\langle \xi - \mathbf{e}_1/\epsilon\rangle^s}\; k_s(\xi - \mathbf{e}_1/\epsilon)\; \Big\|_{L^2(\mathbb{R}^d)} \;\le\; C_s\, |t|\, \epsilon^s \big\|f\big\|_{H^s(\mathbb{R}^d)}\,.$$

The small frequency contribution is negligable in the limit $\epsilon \to 0$. It is removed with a cutoff as above and then the analysis away from $\xi = 0$ proceeds by decomposition into plane wave as in the case with $u_t(0) = 0$.

§1.4. A cautionary example in geometric optics.

A typical science text discussion of a mathematics problem involves simplifying the underlying equations. The usual criterion applied is to ignore terms which are small compared to other terms in the equation. It is striking that in many of the problems treated under the rubric of geometric optics, such an approach can lead to completely innacurate results. It is an example of an area where more careful mathematical consideration is not only useful but necessary.

Consider the initial value problems

$$\partial_t u^\epsilon + \partial_x u^\epsilon + u^\epsilon \;=\; 0\,, \qquad u^\epsilon\big|_{t=0} = a(x)\cos(x/\epsilon)\,,$$

in the limit $\epsilon \to 0$. The function a is assumed to be smooth and to vanish rapidly as $|x| \to \infty$ so the initial value has the form of wave packet. The initial value problem is uniquely solvable and the solution depends continuously on the data. The exact solution of the general problem

$$\partial_t u + \partial_x u + u \;=\; 0\,, \qquad u\big|_{t=0} = f(x)\,,$$

is $u(t,x) = e^{-t} f(x-t)$ so the exact solution u^ϵ is

$$u^\epsilon(t,x) \ = \ e^{-t}\, a(x-t)\, \cos((x-t)/\epsilon)\,.$$

In the limit as $\epsilon \to 0$ one finds that both $\partial_t u^\epsilon$ and $\partial_x u^\epsilon$ are $O(1/\epsilon)$ while $u^\epsilon = O(1)$ is negligibly small in comparison. Dropping this small term leads to the simplified equation for an approximation v^ϵ,

$$\partial_t v^\epsilon + \partial_x v^\epsilon \ = \ 0\,, \qquad v^\epsilon\big|_{t=0} = a(x)\cos(x/\epsilon)\,.$$

The exact solution is

$$v^\epsilon(t,x) \ = \ a(x-t)\cos\big((x-t)/\epsilon\big)\,,$$

which misses the exponential decay. It is **not** a good approximation.

§1.5. The law of reflection.

Consider the wave equation $\Box u = 0$ in the half space $\mathbb{R}^d_- := \{x_1 \le 0\}$. At $\{x_1 = 0\}$ a boundary condition is required. The condition encodes the physics of the interaction with the boundary. For simplicity we consider the familiar Dirichlet condition

$$u(t,x)\big|_{x_1=0} = 0\,. \tag{1.5.1}$$

Cauchy data are prescribed,

$$u(0,x) = f\,, \qquad u_t(0,x) = g\,, \quad \text{for} \quad x_1 \le 0\,. \tag{1.5.2}$$

If the data are supported in a compact subset of $\mathbb{R}^d_-$ then, for small time the support of the solution does not meet the boundary. When waves hit the boundary they are reflected. The goal of this section is to describe this reflection process.

Uniqueness of solutions and finite speed of propagation for (1.5.1)-(1.5.2) are both consequences of a local energy identity. A function is a solution if and only if the real and imaginary parts are solutions. Thus it suffices to treat the real case for which

$$u_t\,\Box u \ = \ \partial_t e - \sum_{j\ge 1} \partial_j(u_t \partial_j u)\,, \qquad e := \frac{u_t^2 + |\nabla_x u|^2}{2}\,.$$

Denote by Γ a backward light cone

$$\Gamma \ := \ \Big\{(t,x) \ : \ |x - \underline{x}|^2 < \underline{t} - t\Big\}$$

and by $\tilde\Gamma$ the part in $\{x_1 < 0\}$,

$$\tilde\Gamma \ := \ \Gamma \ \cap \ \big\{x_1 < 0\big\}\,.$$

For any $0 \le s < \underline{t}$,

$$\tilde\Gamma_s \ := \ \tilde\Gamma \ \cap \ \big\{t = s\big\}\,.$$

Both uniqueness and finite speed follow from the following energy estimate.

Proposition 1.5.1. *If u is a smooth solution of (1.5.1)-(1.5.2), then for $0 < t < \underline{t}$,*

$$\phi(t) \ := \ \int_{\tilde\Gamma_t} e(t,x)\ dx$$

is a nonincreasing function of t.

Proof. Translating the time if necessary it suffices to show that for $s > 0$, $\phi(s) \le \phi(0)$. In the identity

$$0 \ = \ \int_{\tilde\Gamma \cap \{0 \le t \le s\}} u_t\, \Box u\ dt\, dx\,.$$

Integrate by parts to find integrals over four distinct parts of the boundary. The tops and bottoms contribute $\phi(t)$ and $-\phi(0)$ respectively. The intersection of $\tilde\Gamma_s$ with $x_1 = 0$ yields

$$\int_{\tilde\Gamma_s \cap \{x_1 = 0\}} u_t\ \partial_1 u\ dt\, dx_2\ \ldots\ dx_d\,.$$

The Dirichlet condition implies that $u_t = 0$ on this boundary so the integral vanishes. The contribution of the sides $|x - \underline{x}| = \underline{t} - t$ yield an integral of

$$n_0\, e \ + \ \sum_{j=1}^{d} n_j\, u_t\, \partial_j u\,,$$

where $(n_0, n_1, n_2, \ldots, n_d)$ is the outward unit normal. Then

$$n_0 \ = \ \big(\sum_{j=1}^{d} n_j^2\big)^{1/2} \ = \ \frac{1}{\sqrt 2}\,, \qquad \big|\sum_{j=1}^{d} n_j\, u_t\, \partial_j u\big| \ \le \ \frac{1}{\sqrt 2}\ |u_t||\nabla_x u| \ \le \ \frac{1}{\sqrt 2}\, e\,.$$

Thus the integrand from the contributions of sides is nonnegative, so the integral over the sides is nonnegative.
Combining yields

$$\int_{\tilde\Gamma \cap \{0 \le t \le s\}} u_t\, \Box u\ dt\, dx \ \ge \ \phi(t) - \phi(0)\,,$$

and the estimate follows. ∎

We construct and analyse solutions below.

§1.5.1. The method of images.

Proposition 1.5.2. i. *If u is a smooth solution of $\Box u = 0$ satisfying (1.5.1), then*

$$\forall n \ge 0, \quad \frac{\partial^{2n} u}{\partial^{2n} x_1}\bigg|_{x_1 = 0} \ = \ \frac{\partial^{2n} u_t}{\partial^{2n} x_1}\bigg|_{x_1 = 0} \ = \ 0\,. \tag{1.5.3}$$

ii. *Conversely if f, g are smooth functions on $\{x_1 \le 0\}$ satisfying*

$$\forall n \ge 0, \qquad \frac{\partial^{2n} f}{\partial^{2n} x_1}\bigg|_{x_1=0} \;=\; \frac{\partial^{2n} g}{\partial^{2n} x_1}\bigg|_{x_1=0} \;=\; 0\,, \tag{1.5.4}$$

then there is a smooth solution given by the restriction to $x_1 \le 0$ of the solution of the wave equation on $\mathbb{R}^{1+d}$ whose Cauchy data are equal to the odd extensions of f, g.

Proof. i. First prove $\partial_1^{2n} u|_{x_1=0} = 0$ by induction on n. The case $n = 0$ is (1.5.1).

From this case it follows that u_{tt} and $\partial_j^2 u$ with $j > 1$ vanish at $x_1 = 0$. The equation $\Box u = 0$ then shows that $\partial_1^2 u$ vanishes on $\{x_1 = 0\}$, proving the case $n = 1$.

If the case $k \ge$ is known, apply the case k to the solution $\partial_1^2 u$ to prove the case $k + 1$.

The assertions concerning u_t follow since u_t itself is a solution.

Turn next to **ii.** Given Cauchy data (1.5.2), define odd extensions $\tilde f, \tilde g$ to all of $\mathbb{R}^d$ as follows. Let

$$\tilde x \;:=\; (-x-1, x_2, \ldots, x_d)\,, \qquad \tilde\xi \;=\; (-\xi_1, \xi_2, \ldots, x_d)\,.$$

For $x_1 > 0$, define

$$\tilde f(x_1, x_2, \ldots x_d) \;:=\; -f(-x_1, x_2, \ldots, x_d)\,,$$

and

$$\tilde g(x_1, x_2, \ldots x_d) \;:=\; -g(-x_1, x_2, \ldots, x_d)\,.$$

If f and g are smooth on $\{x_1 \le 0\}$ then (1.5.4) is a necessary and sufficient condition in order that $\tilde f$ and $\tilde g$ be smooth on $\mathbb{R}^d$.

Define $\tilde u$ to be the solution of the Cauchy problem for $\Box$ with the smooth initial data $\tilde f, \tilde g$. Then $\tilde u$ is a smooth odd function of x_1. In particular, u satisfies the Dirichlet condition at $x_1 = 0$. Therefore, the restriction of $\tilde u$ to $x_1 \le 0$ is a smooth solution of the mixed initial boundary value problem. ∎

§**1.5.2. The plane wave derivation.**

In many texts you will find a derivation which goes as follows.

Begin with the plane wave solutions

$$e^{i(x.\xi + t\tau)}\,, \qquad \xi \in \mathbb{R}^d, \quad \tau = \mp|\xi|\,.$$

Since u is everywhere of modulus one, no solution of this sort can satisfy the Dirichlet boundary condition.

Seek a solution of the initial boundary value problem which is a sum of two plane waves,

$$e^{i(x.\xi - t|\xi|)} \;+\; A\, e^{i(x.\eta + t\sigma)}\,, \qquad A \in \mathbb{C}\,.$$

In order that the solutions satisfy the wave equation one must have $\sigma^2 = |\eta|^2$. In order that the plane waves sum to zero at $x_1 = 0$ it is necessary and sufficient that $\eta' = \xi'$, $\sigma = -|\xi|$, and $A = -1$. Since $\sigma^2 = |\eta|^2$ it follows that $|\eta| = |\xi|$ so

$$\eta = (\pm\xi_1, \xi_2, \ldots, \xi_d)\,.$$

The sign + yields the solution $u = 0$. The sign minus yields the interesting solution

$$e^{i(x.\xi - t|\xi|)} \; - \; e^{i(\tilde x.\xi - t|\xi|)}$$

which is exactly the odd part of $e^{i(x.\xi-t|\xi|)}$.

The textbook interpretation of the solution with $\tau = -|\xi|$ and $\xi_1 > 0$ is that $e^{i(x.\xi-t|\xi|)}$ is a plane wave approaching the boundary $x_1 = 0$, and $e^{i(\tilde x.\xi-t|\xi|)}$ moves away from the boundary. The first is an incident wave and the second is a reflected wave. The factor $A = -1$ is the reflection coefficient.

Both waves are of infinite extent and of modulus one everywhere in space time. They have finite energy density but infinite energy. They both meet the boundary at all times. It is questionable to think of either one as incoming or reflected. The next subsection shows that there are localized waves which are clearly incoming and reflected waves with the property that when they interact with the boundary the local behavior resembles the plane waves. So, the plane wave computation is accurate.

For more general mixed initial boundary value problems, there are other wave forms which need to be included. The key is that solutions of the form $e^{i(x.\xi+t\tau)}$ are acceptable in $x_1 < 0$ for ξ', τ real and $\operatorname{Im}\xi_1 \le 0$. When $\operatorname{Im}\xi_1 < 0$ the associated waves are localized near the boundary. The Rayleigh waves in elasticity are a classic example. They carry the devastating energy of earth quakes. The reader is referred to Benzoni-Gavage - Serre, Chazarain-Piriou, Taylor (Pseudodifferential), Hormander v.II, Sakamoto, for a more thorough discussion.

§1.5.3. Reflected high frequency wave packets.

Consider solutions which for small time are equal to high frequency solutions from §1.3,

$$u^\epsilon \;\sim\; e^{i(x.\xi - t|\xi|)/\epsilon}\,\Big(a_0(t,x) + \epsilon\, a_1(t,x) + \cdots\Big)\,, \tag{1.5.5}$$

with

$$\xi = (\xi_1, \xi_2, \ldots, \xi_d)\,, \qquad \xi_1 > 0\,.$$

Then $a_0(t,x) \;=\; h(x - t\xi/|\xi|)$ is constant on the rays $\underline{x} + t\xi/|\xi|$. If the Cauchy data are supported in a set ω then the amplitudes a_j are supported in the tube of rays

$$\mathcal{T} \;:=\; \Big\{(t,x)\;:\; x = \underline{x} + t\xi/|\xi|, \quad \underline{x} \in \mathcal{O}\Big\}\,. \tag{1.5.6}$$

The method of images determines the continuation in time of this incoming wave. Define v^ϵ to be the image solution

$$v^\epsilon(t, x_1, x_2, \ldots, x_d) \;:=\; -u^\epsilon(t, -x_1, x_2, \ldots, x_d)\,.$$

The solution of the Dirichlet problem is then equal to the restriction of $u^\epsilon + v^\epsilon$ to $\mathbb{R}^d_-$.

Then

$$\tilde v^\epsilon \;=\; -\,e^{i(\tilde x.\xi - t)/\epsilon}\, h(\tilde x - t\xi) \;+\; \text{h.o.t} \;=\; -\,e^{i(\tilde x.\xi - t)/\epsilon}\, \tilde h(x - t\tilde\xi) \;+\; \text{h.o.t}\,.$$

To leading order, $u^\epsilon + v^\epsilon$ is equal to

$$e^{i(x.\xi - t)/\epsilon}\ h(x - t\xi)\ -\ e^{i(\tilde{x}.\xi - t)/\epsilon}\ \tilde{h}(x - t\tilde{\xi})\,. \tag{1.5.7}$$

The wave represented by u^ϵ has leading term which moves with velocity $\xi/|\xi|$. The wave corresponding to v^ϵ has leading term with velocity $\tilde{\xi}/|\tilde{\xi}|$. Recall that $\tilde{\xi}$ comes from ξ by reversing the first component. At the boundary $x_1 = 0$, the tangential components of $\xi/|\xi|$ and $\tilde{\xi}/|\tilde{\xi}|$ are equal and their normal components are opposite. The directions are related by the standard law that the angle of incidence equals the angle of reflection. The amplitude of the reflected wave v^ϵ on the reflected ray is equal to -1 time the amplitude of the incoming wave u^ϵ on the incoming wave. This is summarized by the statement that the reflection coefficient is equal to -1.

Suppose that $\underline{t}, \underline{x}$ is a point on the boundary and $\mathcal{O}$ in a neighborhood of size large compared to the wavelength ϵ and small compared to the scale on which h varies. Then, on $\mathcal{O}$, the solution is approximately equal to

$$e^{i(x.\xi - t)/\epsilon}\ h(\underline{x} - \underline{t}\xi/|\xi|)\ -\ e^{i(\tilde{x}.\xi - t)/\epsilon}\ \tilde{h}(\underline{x} - \underline{t}\tilde{\xi}/|\tilde{\xi}|)\,.$$

This recovers the reflected plane waves of §1.5.2. An observer on such an intermediate scale sees the structure of the plane waves. Thus, even though the plane waves are completely nonlocal, the asymptotic solutions of geometric optics shows that they predict the local behavior at points of reflection.

An analogous computation for the Neumann boundary condition using *even* mirror reflection in $x_1 = 0$ shows that for that boundary condition the reflection coefficient is equal to 1.

Proposition 1.5.3. i. *If u is a smooth solution of*

$$\Box u = 0 \quad \text{in} \quad x_1 \le 0\,, \qquad \partial_1 u|_{x_1=0} = 0\,, \tag{1.5.8}$$

then

$$\forall n \ge 0, \quad \frac{\partial^{2n+1} u}{\partial^{2n+1} x_1}\bigg|_{x_1=0} \ = \ \frac{\partial^{2n+1} u_t}{\partial^{2n+1} x_1}\bigg|_{x_1=0} \ = \ 0\,. \tag{1.5.9}$$

ii. *Conversely if f, g are smooth functions on $\{x_1 \le 0\}$ satisfying*

$$\forall n \ge 0, \quad \frac{\partial^{2n+1} f}{\partial^{2n+1} x_1}\bigg|_{x_1=0} \ = \ \frac{\partial^{2n+1} g}{\partial^{2n+1} x_1}\bigg|_{x_1=0} \ = \ 0\,, \tag{1.5.10}$$

then there is a smooth solution of (1.5.8) given by the restriction to $x_1 \le 0$ of the solution of the wave equation on $\mathbb{R}^{1+d}$ whose Cauchy data are equal to the even extensions of f, g.

Exercise. Prove the Proposition.

Exercise. Prove uniqueness of solutions by the energy method. **Hint.** Use the local energy identity.

Exercise. Verify the assertion concerning the reflection coefficient by considering the behavior in the future of a solution which near $t = 0$ is a high frequency asymptotic solution approaching the boundary.

§1.6. Snell's law of refraction.

Refraction is the bending of waves as they pass through media whose propagation speeds vary from point to point. The simplest situation is when media with different speeds occupy half spaces, for example $x_1 < 0$ and $x_1 > 0$. The classical physical situations are when light passes from air to water or from air to glass.

A simplified model with the same geometry is

$$u_{tt} - \Delta\, u = 0 \quad \text{in} \quad x_1 < 0\,, \qquad u_{tt} - c^2\, \Delta\, u = 0 \quad \text{in} \quad x_1 > 0\,, \qquad 0 < c < 1\,. \tag{1.6.1}$$

In $x_1 < 0$ the speed is equal to 1 which is greater than the speed c in $x > 0$.

A transmission condition is required at $x_1 = 0$ to encode the interaction of waves with the interface. We analyse the condition which requires that u and $\partial_1 u$ are continuous at $x_1 = 0$. The partial differential equations then imply that $\partial_1^2 u$ is discontinuous at this plane. The physical conditions for Maxwell's Equations at an air-water or air-glass interface can be analysed in the same way. In that case, the coefficient that is discontinuous at the interfaces is the dielectric constant.

Define

$$x' \;:=\; (x_2, \dots, x_d)\,, \qquad \xi' \;:=\; (\xi_2, \dots, \xi_d)\,.$$

Seek solutions of (1.6.1) subject to

$$u(t, 0-, x') \;=\; u(t, 0+, x')\,, \qquad \partial_1 u(t, 0-, x') \;=\; \partial_1 u(t, 0+, x')\,.$$

Denote by square brackets the jump

$$\big[u\big](t, x') \;:=\; u(t, 0+, x') - u(t, 0-, x')\,.$$

The transmission condition is then

$$\big[u\big] \;=\; 0\,, \qquad \big[\partial_1 u\big] \;=\; 0\,. \tag{1.6.2}$$

Define

$$\gamma(x) \;:=\; \begin{cases} 1 & \text{when} \quad x_1 > 0 \\ c^{-2} & \text{when} \quad x_1 < 0\,, \end{cases} \qquad e(t, x) \;:=\; \frac{\gamma\, u_t^2 + |\nabla_x u|^2}{2}\,,$$

From (1.6.1) it follows that solutions suitably small at infinity satisfy

$$\partial_t \int_{x_1 < 0} e\; dx \;=\; \int u_t(t, 0-, x')\; \partial_1 u(t, 0+, x')\; dx'\,,$$

$$\partial_t \int_{x_1 > 0} e\; dx \;=\; -\int u_t(t, 0+, x')\; \partial_1 u(t, 0+, x')\; dx'\,.$$

The transmission condition guarantees that the terms on the right compensate exactly so

$$\partial_t \int_{\mathbb{R}^3} e \, dx \;=\; 0\,. \tag{1.6.3}$$

This suffices to prove uniqueness of solutions. A localized argument as in §1.5.1, shows that signals travel at most at speed one.

Exercise. Prove this finite speed result.

A function $u(t,x)$ is called **piecewise smooth** if its restriction to $x_1 < 0$ (resp. $x_1 > 0$) has a C^∞ extension to $x_1 \le 0$ (resp. $x_1 \ge 0$). The Cauchy data of piecewise smooth solutions must be piecewise smooth (with the analogous definition for functions of x only). They must, in addition, satisfy a sequence of compatibility conditions.

Propostion 1.6.1. *If u is a piecewise smooth solutions u of the transmission problem, then*

$$\Delta^j\{u,u_t\}(t,0-,x_2,x_3) \;=\; (c^2\Delta)^j\{f,g\}(t,0+,x_2,x_3)\,,$$

$$\Delta^j\partial_1\{u,,u_t\}(t,0-,x_2,x_3) \;=\; (c^2\Delta)^j\partial_1\{f,g\}(t,0+,x_2,x_3)\,.$$

ii. *Conversely, if the piecewise smooth f,g satisfy for all $j\ge 0$,*

$$\Delta^j\{f,g\}(0-,x_2,x_3) \;=\; (c^2\Delta)^j\{f,g\}(0+,x_2,x_3)\,, \tag{1.6.4}$$

$$\Delta^j\partial_1\{f,g\}(0-,x_2,x_3) \;=\; (c^2\Delta)^j\partial_1\{f,g\}(0+,x_2,x_3)\,. \tag{1.6.5}$$

then there is a piecewise smooth solution with these Cauchy data.

Proof. If u is a piecewise smooth solution then so is $\partial_t^j u$ for any j. The key is that the transmission condition (1.6.2) and be differentiated in t so

$$\big[\partial_t^j u\big] \;=\; 0\,, \qquad \big[\partial_t^j\partial_1 u\big] \;=\; 0\,. \tag{1.6.6}$$

The case $j=1$ yields the necessary condition

$$\big[g\big] \;=\; 0\,, \qquad \big[\partial_1 g\big] \;=\; 0\,.$$

For the higher orders, compute with $k\ge 1$,

$$\partial_t^{2k}u\big|_{t=0} \;=\; \begin{cases} \Delta^k u & \text{when} \quad x_1<0 \\ (c^2\Delta)^k u & \text{when} \quad x_1>0, \end{cases}$$

$$\partial_t^{2k-1}u\big|_{t=0} \;=\; \begin{cases} \Delta^k u & \text{when} \quad x_1<0 \\ (c^2\Delta)^k u & \text{when} \quad x_1>0. \end{cases}$$

Thus, the transmission conditions (1.6.6) prove **i.**

The proof of **ii.** is technical, interesting, and omitted. One can construct solutions using finite differences almost as in §2.2. The shortest existence proof to state uses the spectral theorem for self adjoint operators. * The general regularity theory for such transmission problems can be obtained by folding them to a boundary value problem and using the results of [Rauch-Massey, Sakamoto]. ∎

Suppose

$$\xi \ \in \mathbb{R}^d, \qquad |\xi| = 1\,, \qquad \xi_1 > 0\,,$$

and consider an incoming high frequency asymptotic solution in $\{x_1 < 0\}$,

$$I^\epsilon \ \sim \ e^{i(x.\xi - t)/\epsilon}\, a(\epsilon, t, x)\,, \qquad a(\epsilon, t, x) \ \sim \ a_0(t,x) + \epsilon\, a_1(t,x) + \cdots\,, \tag{1.6.5}$$

with initial support compact in $\{x_1 < 0\}$. Since the incoming waves are smooth and initially vanish identically on a neighborhood of the interface $\{x_1 = 0\}$, the compatibilities are satisfied and there is a piecewise smooth solution u defined on $\mathbb{R}^{1+d}$.

Seek an asymptotic solution which at $\{t = 0\}$ is equal to this incoming wave.

Experience suggests that at the interface there will be a reflected wave in $x_1 < 0$ and a refracted wave in $x_1 > 0$. By uniqueness it suffices to construct such a solution to verify the expectation.

Seek the reflected wave in $x_1 \ge 0$ in the form

$$R^\epsilon \ \sim \ e^{i(x.\tilde\xi - t)/\epsilon}\, b(\epsilon, t, x)\,, \qquad b(\epsilon, t, x) \ \sim \ b_0(t,x) + \epsilon\, b_1(t,x) + \cdots\,. \tag{1.6.6}$$

---

* For those with sufficient background, the Hilbert space is $\mathcal{H} := L^2(\mathbb{R}^d\,;\, \gamma\, dx)$.

$$D(\mathcal{A}) \ := \ \Big\{ w \in H^2(\mathbb{R}^d_+) \cap H^2(\mathbb{R}^d_-) \ : \ [w] = [\partial_1 w] = 0 \Big\}\,,$$

$$\mathcal{A}w := \Delta w \quad \text{in} \quad x_1 < 0, \qquad \mathcal{A}w := c^2 \Delta \quad \text{in} \quad x_1 > 0\,.$$

Then,

$$(\mathcal{A}u, v)_{\mathcal{H}} \ = \ (u, \mathcal{A}v)_{\mathcal{H}} \ = \ -\int \nabla u . \nabla v \ dx\,,$$

so $-\mathcal{A} \ge 0$. The elliptic regularity theorem implies that $\mathcal{A}$ is self adjoint. The regularity theorem is proved, for example, by the methods in Chapter 10 of [Rauch, book]. The solution of the initial value problem is

$$u \ = \ \cos t\sqrt{-\mathcal{A}}\ f \ + \ \frac{\sin t\sqrt{-\mathcal{A}}}{\sqrt{-\mathcal{A}}}\ g\,.$$

For piecwise H^∞ data, the sequence of compatibilities is equivalent to the data belonging to $\cap_j D(\mathcal{A}^j)$.

Seek the transmitted wave in $x_1 > 0$ in the form

$$T^\epsilon \ \sim \ e^{i(x.\eta - t\tau)/\epsilon}\, d(\epsilon,t,x)\,, \qquad d(\epsilon,t,x) \ \sim \ d_0(t,x) + \epsilon\, d_1(t,x) + \cdots\,, \quad \eta_1 > 0\,. \quad (1.6.7)$$

In order that this be an approximate solution moving away from the interface one must have

$$\tau^2 = c^2|\eta|^2\,, \qquad \tau/\eta_1 > 0\,.$$

Summarizing seek

$$u^\epsilon \ = \ \begin{cases} I^\epsilon + R^\epsilon & \text{in} \quad x_1 < 0 \\ \quad T^\epsilon & \text{in} \quad x_1 > 0 \end{cases}\,.$$

The continuity required at $x_1 = 0$ forces

$$e^{i(x'.\xi' - t)/\epsilon}\,\big(a(\epsilon,t,0,x') + b(\epsilon,t,0,x')\big) \ = \ e^{i(x'.\eta' - t\tau)/\epsilon}\, d(\epsilon,t,0,x')\,. \qquad (1.6.8)$$

For there to be solutions with nonzero leading amplitudes the exponential factors must be equal. The equality of exponentials holds if and only if

$$\eta' \ = \ \xi' \quad \text{and} \quad \tau = 1\,.$$

The sign convention on $\tau/\eta_1 > 0$ forces

$$\eta_1 = \big(c^{-2} - |\xi'|^2\big)^{1/2}$$

completely determining η.

Once η is determined the continuity of u and $\partial_1 u$ hold if and only if at $x_1 = 0$ one has

$$a \ + \ b \ = \ d\,, \qquad \text{and} \qquad \frac{i\xi_1}{\epsilon}a + \partial_1 a \ - \ \frac{i\xi_1}{\epsilon}b + \partial_1 b = \frac{i\xi_1}{\epsilon}d + \partial_1 d\,. \qquad (1.6.9)$$

The first of these relations yields

$$\Big(a_j \ + \ b_j \ - \ d_j\Big)_{x_1=0} = 0, \qquad j = 0,1,2,\dots\,, \qquad (1.6.10)$$

The second relation in (1.6.9) is expanded in powers of ϵ. The coefficients of ϵ^j must match for all all $j \geq -1$. The leading order is ϵ^{-1} and yields

$$\big(a_0 - b_0 - d_0\big)_{x_1=0} = 0\,. \qquad (1.6.11)$$

Since a_0 is known, the $j = 0$ equation from (1.6.10) together with (1.6.11) suffice to determine b_0, d_0 at $x_1 = 0$. The amplitude b_0 (resp. d_0) is constant on rays with speed $\tilde{\xi}$ (resp. η). Thus the leading amplitudes are determined throughout the half spaces on which they are defined.

Once these leading terms are known the ϵ^0 term from the second equation in (1.6.9) shows that on $x_1 = 0$,

$$a_1 - b_1 - d_1 = \text{known}\,.$$

Note that a_1 is also known so that together with the case $j = 2$ from (1.6.10) this suffices to determine b_1, d_1 on $x_1 = 0$. Each satisfies a transport equation along rays which is the analogue of (1.3.12), so from b_0, d_0 and the data on $x_1 = 0$ they are knowm everywhere. The higher order correctors are detemined analogously.

Once the b_j, d_j are determined, one can choose b, c with those Taylor coefficients and so that (1.6.9) is exactly satisfied. The function u^ϵ is then an infinitely accurate approximate solution in the sense that there is an exact solution which differs from the approximation by $O(\epsilon^N)$ for all $N > 0$.

If θ_i and θ_r are the angles of incidence and refraction then

$$\sin\theta_i \;=\; \frac{|\xi'|}{|\xi|}, \qquad \text{and}, \qquad \sin\theta_r \;=\; \frac{|\eta'|}{|\eta|} \;=\; \frac{|\xi'|}{c|\xi|}\,.$$

Therefore

$$\frac{\sin\theta_i}{\sin\theta_r} \;=\; \frac{1}{c}\,,$$

is independent of θ_i. The high frequency asymptotic solutions explain Snell's law. This is the last of the three basic laws of geometric optics.

Note in passing on a neighborhood $\underline{t}, \underline{x} \in \{x_1 = 0\}$ which is small compared to the scale on which a, b, c vary an large compared to ϵ, the solution resembles three interacting plane waves. In science texts one usually computes for which such triples the transmission condition is satisfied in order to find Snell's law. The asymmptotic solutions of geometric optics show how to overcome the criticism that the plane waves are infinitely diffuse so cannot reasonable be viewed as either incoming or outgoing.

For a more complete discussion of refection and refraction see [Taylor, Pseudodifferential, Benzoni-Gavage-Serre].